Name: Solutions

Math 4200 Quiz week 11 November 4, 2020

1a) The function $f(z) = \dfrac{e^{\frac{1}{z}}}{1-z}$ is analytic for $0 < |z| < 1$. Find its residue at $z_0 = 0$.

coeff of $\frac{1}{z}$

(6 points)

$$e^{\frac{1}{z}} \cdot \frac{1}{1-z} = \left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\right)\left(1 + z + z^2 + \cdots\right)$$

$\frac{1}{z}$ coeff: $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = \boxed{e-1}$

often easier to write series out long-form rather than with $\sum$ notation.

1b) Let γ be the circle of radiius $\frac{1}{2}$ centered at the origin. For the function $f(z)$ in part (a), find

$$\int_\gamma f(z)\, dz.$$

$$\oint_\gamma f(z)dz = 2\pi i \,\mathrm{Res}(f;0) = 2\pi i\,(e-1)$$

(4 points)

Math 4200
Friday November 6

- 4.1 Calculating residues at isolated singularities
- 4.2 The Residue Theorem(s) - tie together and extend previous contour integration tricks.

Announcements: numbering typo on HW due today - last problem should be w10.5

any hw ?'s ?

3.3.20 b) Let f entire
Consider $f\left(\frac{1}{z}\right)$.

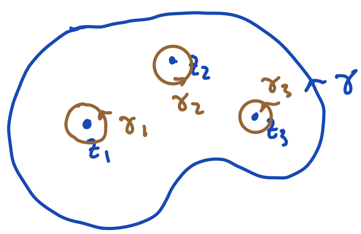
Midterm a week from today. thru § 4.2.

* for a practice exam ~ look at fall 2011 exam 2.

Residue Theorem (Replacement Theorem version): Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \dots z_k\} \subseteq A$ Let γ be a simple closed contour in A which contains none of the singularities, and which bounds a subregion B containing some of the singularities, in the counterclockwise direction. Then

$$\int_\gamma f(z)\, dz = \boxed{2\pi i \sum_{z_j \in B} Res(f, z_j)}$$

proof: Use The section 2.2 Replacment Theorem for domains with holes together Laurent series for f at the singularities, and the diagram below. Notice our notation for residues...



$$\oint_\gamma f(z)dz = \sum_{z_j \in B} \int_{\gamma_j} f(z)dz$$

on γ_j use Laurent series for f on $D(z_j;r)\setminus\{z_j\}$

$$f(z) = \sum_{n=0}^{\infty} a_{nj}(z-z_j)^n + \sum_{m=1}^{\infty} b_{mj}(z-z_j)^{-m}$$

conv. unif on γ_j

$$\int_{\gamma_j} f(z)dz = \int_{\gamma_j} \Sigma_1 + \Sigma_2\, dz = \sum \int_{\gamma_j}$$

FTC $\Rightarrow$ all terms except $\frac{1}{z-z_j}$ are zero

and that one gives us $\boxed{b_{-1j}\; 2\pi i}$

Exercise: Show - just for the practice - that this theorem includes as special cases

(a) Cauchy's theorem that $\int_\gamma f(z)\, dz = 0$ if f is analytic in A.

RHS = 0

no singular pts inside γ.

(b) CIF $\boxed{f(z_0) = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(z)}{z - z_0}\, dz}$ if z_0 is inside γ and f is analytic in A.

$\Rightarrow g(z) = \dfrac{f(z)}{z-z_0}$ has one singular pt. z_0.

So we would like systematic ways to compute residues. There's a page in the book...

near z_0

$$g(z) = \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots}{(z-z_0)}$$

$$= \frac{a_0}{(z-z_0)} + a_1 + a_2(z-z_0)^2 + \dots$$

$$Res(g; z_0) = a_0 = f(z_0)$$

Res Thm

$$\int_\gamma g(z)dz = 2\pi i\, Res(g; z_0) = 2\pi i\, f(z_0)$$

§4.1 how to compute residues ~ it all goes back to power series tricks.

Prop 4.1.7. $f(z) = \frac{g(z)}{h(z)}$ $g(z_0) \neq 0$. $h(z)$ zero of order $k-1$, i.e. $h^{(k)}(z_0)$ is 1st non-zero deriv.

the residue at z_0, $\mathrm{Res}(g/h; z_0)$ *is given by*

$$\mathrm{Res}(g/h;z_0) = \left[\frac{k!}{h^{(k)}(z_0)}\right]^k \times \begin{vmatrix} \frac{h^{(k)}(z_0)}{k!} & 0 & 0 & \cdots & 0 & g(z_0) \\ \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & 0 & \cdots & 0 & g^{(1)}(z_0) \\ \frac{h^{(k+2)}(z_0)}{(k+2)!} & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{h^{(k)}(z_0)}{k!} & \cdots & 0 & \frac{g^{(2)}(z_0)}{2!} \\ \vdots & \vdots & \vdots & & & \vdots \\ \frac{h^{(2k-1)}(z_0)}{(2k-1)!} & \frac{h^{(2k-2)}(z_0)}{(2k-2)!} & \frac{h^{(2k-3)}(z_0)}{(2k-3)!} & \cdots & \frac{h^{(k+1)}(z_0)}{(k+1)!} & \frac{g^{(k-1)}(z_0)}{(k-1)!} \end{vmatrix},$$

where the vertical bars denote the determinant of the enclosed $k \times k$ *matrix.*

Table 4.1.1 Techniques for Finding Residues

In this table g and h are analytic at z_0 and f has an isolated singularity. The most useful and common tests are indicated by an asterisk.

	Function	Test	Type of Singularity	Residue at z_0
✓ 1.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$	removable	0
✓ *2.	$\frac{g(z)}{h(z)}$	g and h have zeros of same order → N	removable	0
*3.	$f(z)$	$\lim_{z\to z_0}(z-z_0)f(z) = 0$ exists and is $\neq 0$	simple pole	$\lim_{z\to z_0}(z-z_0)f(z)$
*4.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0) = 0,$ $h'(z_0)\neq 0$	simple pole	$\frac{g(z_0)}{h'(z_0)}$
5.	$\frac{g(z)}{h(z)}$	g has zero of order k, h has zero of order $k+1$	simple pole	$(k+1)\frac{g^{(k)}(z_0)}{h^{(k+1)}(z_0)}$
*6.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0$ $h(z_0) = 0 = h'(z_0)$ $h''(z_0)\neq 0$	second-order pole	$2\frac{g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{[h''(z_0)]^2}$
*7.	$\frac{g(z)}{(z-z_0)^2}$	$g(z_0)\neq 0$	second-order pole	$g'(z_0)$
*8.	$\frac{g(z)}{h(z)}$	$g(z_0) = 0, g'(z_0)\neq 0,$ $h(z_0) = 0 = h'(z_0)$ $= h''(z_0), h'''(z_0)\neq 0$	second-order pole	$3\frac{g''(z_0)}{h'''(z_0)} - \frac{3}{2}\frac{g'(z_0)h^{(iv)}(z_0)}{[h'''(z_0)]^2}$
9.	$f(z)$	k is the smallest integer such that $\lim_{z\to z_0}\phi(z_0)$ exists where $\phi(z) = (z-z_0)^k f(z)$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$
*10.	$\frac{g(z)}{h(z)}$	g has zero of order l, h has zero of order $k+l$	pole of order k	$\lim_{z\to z_0}\frac{\phi^{(k-1)}(z)}{(k-1)!}$ where $\phi(z) = (z-z_0)^k\frac{g}{h}$
11.	$\frac{g(z)}{h(z)}$	$g(z_0)\neq 0, h(z_0) =$ $\ldots = h^{k-1}(z_0)$ $= 0, h^k(z_0)\neq 0$	pole of order k	see Proposition 4.1.7.

2

$$\frac{g(z)}{h(z)} = \frac{a_N(z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots}{b_N(z-z_0)^N + b_{N+1}(z-z_0)^{N+1} + \cdots} = \frac{\cancel{(z-z_0)^N}}{\cancel{(z-z_0)^N}}\underbrace{\left[\frac{a_N + a_{N+1}(z-z_0) + \cdots}{b_N + b_{N+1}(z-z_0) + \cdots}\right]}_{\text{analytic at } z_0}$$

quot. of two analytic fns where denom $\neq 0$ at z_0

h has a zero of order 1 @ z_0

Table entry 4: Let $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z_0) \neq 0,\ h(z_0) = 0,\ h'(z_0) \neq 0$. Prove that f has a pole of order 1, and

- $Res(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}$

pf $\dfrac{g(z)}{h(z)} = \dfrac{\overset{\neq 0}{a_0} + a_1(z-z_0) + \cdots}{b_1(z-z_0) + b_2(z-z_0)^2 + \cdots}$

$= \dfrac{1}{z-z_0}\left[\dfrac{a_0 + a_1(z-z_0) + \cdots}{b_1 + b_2(z-z_0) + \cdots}\right]$ analytic near z_0

$= \dfrac{1}{z-z_0}\left[c_0 + c_1(z-z_0) + \cdots\right]$ $\quad c_0 = \dfrac{a_0}{b_1} = Res(f; z_0)$

$= \dfrac{c_0}{z-z_0} + c_1 + \cdots$ $\quad = \dfrac{g(z_0)}{h'(z_0)}$ ✓.

- *Example:* The table entry above is great for quotient functions in general, as long as $h(z)$ only has zeroes of order 1. So it makes many rational function contour integrals a breeze, without having to use long division and/or partial fractions.

Compute $\boxed{\displaystyle\int_{|z|=3} \frac{z^3}{z^2 - 2z}\,dz}$ or e^z or $\cos 5z$ (And notice that it would be just as easy for an arbitrary analytic function in the numerator.

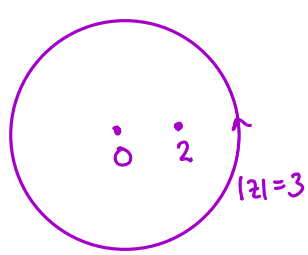
$z_0^3 \qquad z_0^2 - 2z_0 \qquad h'(z_0) = 2z_0 - 2$

Table entry 4: Let $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z_0) \neq 0,\ h(z_0) = 0,\ h'(z_0) \neq 0$. Prove that f has a pole of order 1, and

$\boxed{Res(f, z_0) = \dfrac{g(z_0)}{h'(z_0)}}$

Chptr 2 $\quad z^2 - 2z = z(z-2)$

long div.
parfrac
deformation thm
(replacement thm).



$f(z) = \dfrac{z^3}{z(z-2)} = \dfrac{g(z)}{h(z)}$

Res Thm

$\displaystyle\oint_{|z|=3} \frac{z^3}{z^2-2z}\,dz = 2\pi i\left(Res(f;0) + Res(f;2)\right)$

$= 2\pi i\left[\dfrac{g(0)}{h'(0)} + \dfrac{g(2)}{h'(2)}\right]$

$= 2\pi i\left[0 + \dfrac{8}{4-2}\right] = 8\pi i$

Table entry 7: If $f(z)$ has a pole of order k at z_0, and has the form

$$f(z) = \frac{g(z)}{\left(z - z_0\right)^k} = \frac{a_0 + a_1(z-z_0) + a_2(z-z_0)^2}{(z-z_0)^k}$$

then

$$Res\left(f, z_0\right) = \frac{g^{(k-1)}\left(z_0\right)}{(k-1)!}$$

divide through
coeff of $\frac{1}{z-z_0}$ is a_{k-1}

// Taylor

Example Compute $Res\left(\frac{e^{2z}}{(z-1)^2}; 1\right)$

Table entry 6: Let $f(z) = \dfrac{g(z)}{h(z)}$ where $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$.

Then f has ~~a~~ pole of order 2 and

$$Res(f, z_0) = \frac{2\,g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{g(z_0)h'''(z_0)}{h''(z_0)^2} \qquad !!!$$

(This leads into Prop 4.1.7, your extra credit hw problem.)

$$f(z) = \frac{\overset{\neq 0}{a_0} + a_1(z-z_0) + a_2(z-z_0)^2}{b_2(z-z_0)^2 + b_3(z-z_0)^3 + \cdots}$$

$$f(z) = \frac{1}{(z-z_0)^2}\cdot\left(\begin{matrix}\text{analytic } \& \neq 0\\ \text{at } z_0\end{matrix}\right)$$

$$\underbrace{\left(\frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \cdots\right)}_{f}\underbrace{\left(b_2(z-z_0)^2 + b_3(z-z_0)^3 + \cdots\right)}_{h} = \underbrace{a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots}_{g}$$

equate power coef's.

$(z-z_0)^0$: $\quad c_{-2}b_2 = a_0$

$(z-z_0)^1$: $\quad c_{-2}b_3 + c_{-1}b_2 = a_1$

$$\begin{bmatrix} b_2 & 0 \\ b_3 & b_2 \end{bmatrix}\begin{bmatrix} c_{-2} \\ c_{-1} \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Cramer's rule! $\quad c_{-1} = \dfrac{\begin{vmatrix} b_2 & a_0 \\ b_3 & a_1 \end{vmatrix}}{\begin{vmatrix} b_2 & 0 \\ b_3 & b_2 \end{vmatrix}} = \dfrac{b_2a_1 - b_3a_0}{b_2^2}$

$$= \frac{a_1}{b_2} - \frac{b_3a_0}{b_2^2}$$

$$= \frac{g'(z_0)}{\frac{1}{2}h''(z_0)} - \frac{\frac{1}{3!}h'''(z_0)g(z_0)}{\left(\frac{1}{2!}h''(z_0)\right)^2}$$

$$!! \qquad = \frac{2g'(z_0)}{h''(z_0)} - \frac{2}{3}\frac{h'''(z_0)g(z_0)}{(h''(z_0))^2}$$

Residue Theorem (Deformation Theorem version, more general than the Green's Theorem version.). Let Let f be analytic on a region A, except on a finite set of isolated singularities $\{z_1, z_2, \ldots z_k\} \subseteq A$. Let γ be a closed curve which is homotopic to a point in A. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{k} Res(f, z_j)\, I(\gamma; z_j)$$

Being more general than the Green's Theorem version, this proof is also a bit more complicated. For each isolated singularity z_j we have the Laurent series

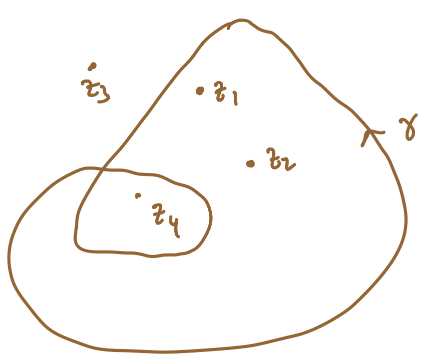
$$S_j(z) = S_{j1}(z) + S_{j2}(z) = \sum_{n=0}^{\infty} a_{jn}(z - z_j)^n + \sum_{m=1}^{\infty} \frac{b_{jm}}{(z - z_j)^m}.$$

Because the z_j are point singularities, the singular part of the series, $S_{j2}(z)$ converges in $\mathbb{C} \setminus \{z_j\}$, and the non-singular part converges for $0 \le |z - z_j| < R_j$ for some positive radius of convergence R_j.

Now consider

$$g(z) := f(z) - \sum_{j=1}^{k} S_{j2}(z).$$

Explain why $g(z)$ has removable singularities at each z_l:



Thus we may consider g to be analytic in A, so since γ is homotopic as closed curves to a point in A,

$$\int_{\gamma} g(z)\ dz = 0.$$

Expand this to get the result!

Math 4200-001
Homework 11
4.1-4.2
Due Wednesday November 11 at 11:59 p.m.
Exam will cover thru 4.2

4.1 1de, 3, 5, 7ab, 9
4.2 2 (Section 2.3 Cauchy's Theorem), 3, 4, 6, 9, 13.

w11.1 (extra credit) Prove Prop 4.1.7, the determinant computation for the residue at an order k pole for $f(z) = \frac{g(z)}{h(z)}$ at z_0, where $g(z_0) \neq 0$. (Hint: it's Cramer's rule for a system of equations.)